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FINITE MINKOWSKI PLANES

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Finite Minkowski planes*)

by

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ABSTRACT

In this paper we give second characterizations of a certain class of finite Minkowski planes.

KEY WORDS & PHRASES: *Minkowski plane, affine plane, nearaffine plane.*

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*) This report will be submitted for publication elsewhere

1. INTRODUCTION

It is well known, see e.g. [5], that with each point of a Minkowski plane there is associated an affine plane, its so-called derived plane. It is the purpose of this paper to show that, under certain additional hypotheses, with each point of a Minkowski plane there is also associated a nearaffine plane, its *residual* plane. In addition we show that the "known" Minkowski plane are characterized by the fact that these nearaffine planes are nearaffine translation planes (see [9]). Using this result a configurational condition is obtained in a completely natural way which characterizes the known Minkowski planes.

2. BASIC CONCEPTS

Let M be a set of *points* and $\mathcal{L}^+$, $\mathcal{L}^-$, $\mathcal{C}$ three collections of subsets of M. The elements of $\mathcal{L} := \mathcal{L}^+ \cup \mathcal{L}^-$ are called *lines* or *generators*, the elements of C are called *circles*. We say that $\mathcal{M} = (M,\mathcal{L}^+,\mathcal{L}^-,\mathcal{C})$ is a *Minkowski plane* if the following axioms are satisfied (cf. [5]):

(M1): $\mathcal{L}^+$ and $\mathcal{L}^-$ are partitions of M.

(M2): $|\ell^+ \cap \ell^-| = 1$ for all $\ell^+ \in \mathcal{L}^+$, $\ell^- \in \mathcal{L}^-$.

(M3): Given any three points no two or a line, there is a unique circle passing through these three points.

(M4): $|\ell \cap c| = 1$ for all $\ell \in \mathcal{L}$, $c \in \mathcal{C}$.

(M5): There exist three points no two of which are on one line.

(M6): Given a circle c, a point $P \in c$ and a point $Q \notin c$, P and Q not on one line, there is a unique circle d such that $P,Q \in d$ and $c \cap d = \{P\}$.

Two points P and Q are called *plus-parallel* (notation $P \parallel_+ Q$) if P and Q are on a line of $\mathcal{L}^+$, *minus-parallel* ($P \parallel_- Q$) if P and Q are on a line of $\mathcal{L}^-$. *Parallel* ($P \parallel Q$) means either $P \parallel_+ Q$ or $P \parallel_- Q$. For $P \in M$, $\epsilon = +,-$ we denote by $[P]_\epsilon$ the unique line in $\mathcal{L}^\epsilon$ incident with P. If P, Q and R are (distinct) nonparallel points, then we denote by (P,Q,R) the unique circle containing P, Q and R. Two circles c and d *touch* in a point P if $c \cap d = \{P\}$.

Fix a point Z and put

$$M_Z := M\backslash([Z]_+ \cup [Z]_-),$$

$$L_Z := \{c^* | c \in \mathcal{C},\ Z \in c\} \cup \{\ell^* | \ell \in \mathcal{L}\backslash\{[Z]_+, [Z]_-\}\},$$

where the * indicates that we have removed the point that the circle or line has in common with $[Z]_+ \cup [Z]_-$. Then $\mathcal{M}_Z := (M_Z, L_Z)$ is an affine plane with pointset M_Z and lineset L_Z (see e.g. [5]). We call $\mathcal{M}_Z$ the *derived plane* with respect to the point Z. We shall only consider finite Minkowski planes, i.e. Minkowski planes with a finite number of points. For finite Minkowski planes (M6) is a consequence of the other axioms (see [5]). It is easily seen that $|\mathcal{L}^+| = |\mathcal{L}^-| = |\ell| = |c| =: n+1$ for all $\ell \in \mathcal{L}$, $c \in \mathcal{C}$. The integer n is called the *order* of the Minkowski plane. Notice that n is also the order of the derived planes $\mathcal{M}_Z$.

Following BENZ [1] we sketch the close relationship between (finite) Minkowski planes and sharply 3-transitive sets of permutations. Let Ω be a finite set, $|\Omega| = n+1 \geq 3$, and G a subset of S^Ω, the symmetric group on Ω, acting sharply triply transitively on Ω.

Define

$$M := \Omega\times\Omega,$$

$$\mathcal{L}^+ := \{\{(\alpha,\beta) |\ \alpha \in \Omega\} |\ \beta \in \Omega\},$$

$$\mathcal{L}^- := \{\{(\alpha,\beta) |\ \beta \in \Omega\} |\ \alpha \in \Omega\},$$

$$\mathcal{C} := \{\{(\alpha,\alpha^g) |\alpha \in \Omega\} |\ g \in G\}.$$

Then $\mathcal{M} := (\Omega,G) := (M,\mathcal{L}^+,\mathcal{L}^-,\mathcal{C})$ is a Minkowski plane of order n. Conversely, every Minkwoski plane can be obtained in this way.

Two Minkwoski planes $\mathcal{M} = (\Omega,G) = (M,\mathcal{L}^+,\mathcal{L}^-,\mathcal{C})$ and $\mathcal{M}' = (\Omega',G') = (M',\mathcal{L}^{+\prime},\mathcal{L}^{-\prime},\mathcal{C}')$ are said to be *isomorphic* if there is a bijection $s\colon M \to M'$ such that

$$\mathcal{L}^s = \mathcal{L}' \quad \text{and} \quad \mathcal{C}^s = \mathcal{C}'.$$

Since s maps the disjoint lines of $\mathcal{L}^+$ onto disjoint lines there are only two possibilities, either $(\mathcal{L}^\varepsilon)^s = \mathcal{L}^\varepsilon$ or $(\mathcal{L}^\varepsilon) = \mathcal{L}^{-\varepsilon}$, $\varepsilon = +,-$. In the first case s is called a *positive isomorphism* in the second case a *negative* *isomorphism.* If s is a positive isomorphism then there exist bijections $a,b\colon \Omega \to \Omega'$ such that $(\alpha,\beta)^s = (\alpha^a,\beta^b)$ for all $\alpha,\beta \in \Omega$, and $G' = a^{-1}Gb$. If s is a negative isomorphism then there exist bijections $a,b\colon \Omega \to \Omega'$ such that $(\alpha,\beta)^s = (\beta^b,\alpha^a)$, and $G' = b^{-1}G^{-1}a$. It follows that we may assume w.l.o.g that $id \in G$.

A (*positive, negative*) automorphism of a Minkowski plane M is a (positive, negative) isomorphism of M onto itself. The automorphism group $Aut(\Omega,G) \le S^{\Omega\times\Omega}$ of the Minkowski plane (Ω,G) is given by

$$Aut(\Omega,G) = \{(a,b) \mid a^{-1}Gb = G\} \cup \{(a,b) \mid a^{-1}Gb = G^{-1}\}\tau$$

where τ is the permutation which sends (α,β) to (β,α).

3. THE RESIDUAL PLANE

Let $M = (M,L^+,L^-,C)$ be a Minkowski plane. Fix a point $Z \in M$ and define $M_Z = M\setminus([Z]_+\cup[Z]_-)$. We have already remarked that the lines $\neq[Z]_+,[Z]_-$ together with the circles which are incident with Z are the lines of an affine plane with pointset M_Z. We shall show that the lines $\neq[Z]_+,[Z]_-$ together with the circles not incident with Z are the lines of a nearaffine plane with the same pointset if suitable conditions are assumed to hold is M.

For each point $P \in M_Z$ we let the points P^+ and P^- be defined by $P^+ := [Z]_+\cap[P]_-$, $P^- := [Z]_-\cap[P]_+$. The restriction of a line ℓ or circle c to M_Z is denoted by $\ell^* := \ell\cap M_Z$ resp. $c^* := c\cap M_Z$. For any two distinct points $P,Q \in M_Z$ we define

$$P \sqcup Q := \begin{cases} \ell^* & \text{iff } P,Q \in \ell \in L, \\ \{P\} \cup (P^+,P^-,Q)^* & \text{iff P and Q are nonparallel.} \end{cases}$$

Since two circles can have at most two points in common it follows that $P \sqcup Q = Q \sqcup P$ if and only if $P \sqcup Q = \ell^*$ for some $\ell \in L$, provided the order n of M is at least 5. The verification of the axioms (L1), (L2) and L(3) (see [9]) is now straightforward. In order to define parallelism we have to require that the following condition holds in M for every point Z.

(A): Let P_1, Q_1, P_2, $Q_2 \in M_Z$ and suppose that P_1 and Q_1, P_2 and Q_2, P_1 and P_2 are nonparallel. If there exists a circle c touching (P_1^+, P_1^-, Q_1) in P_1^- and touching (P_2^+, P_2^-, Q_2) in P_2^+, then there also exists a circle d touching (P_1^+, P_1^-, Q_1) in P_1^+ and touching (P_2^+, P_2^-, Q_2) in P_2^- (see figure 1).

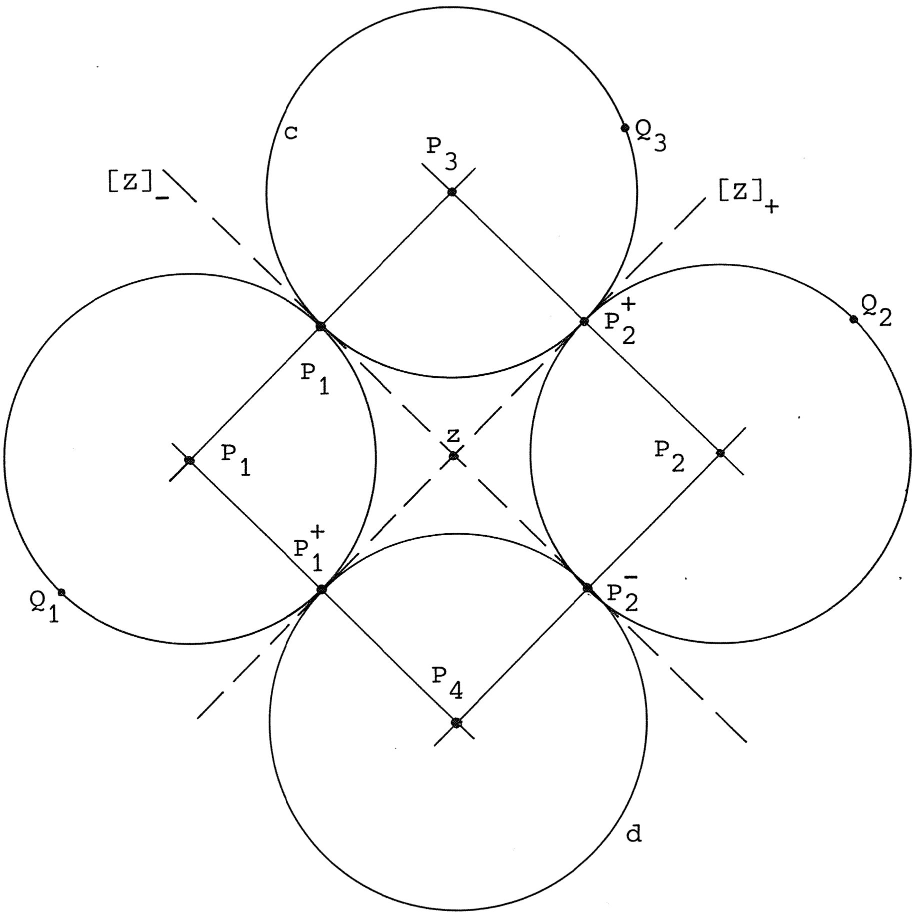

Fig. 1.

In the definition of $P_1 \sqcup Q_1 \parallel P_2 \sqcup Q_2$ we have to distinguish several cases.

Case 1: P_1 and Q_1 parallel, say $P_1 \sqcup Q_1 = \ell_1^*$ for some $\ell_1 \in L^\varepsilon$.

$$P_1 \sqcup Q_1 \parallel P_2 \sqcup Q_2 :\Longleftrightarrow P_2 \sqcup Q_2 = \ell_2^* \quad \text{for some } \ell_2 \in L^\varepsilon.$$

Case 2: P_1 and Q_1 nonparallel, P_1, P_2 parallel, say $P_1, P_2 \in \ell \in L^\varepsilon$. From [9], proposition 3.1, it is clear that we have to define

$$P_1 \sqcup Q_1 \parallel P_2 \sqcup Q_2 :\Longleftrightarrow P_1 \sqcup Q_1 = P_2 \sqcup Q_2 \text{ or } (P_1 \sqcup Q_1) \cap (P_2 \sqcup Q_2) = \emptyset.$$

Case 3: P_1 and Q_1 nonparallel and P_1, P_2 nonparallel. Put $P_3 = [P_1]_+ \cap [P_2]_-$ and $P_4 := [P_1]_- \cap [P_2]_+$ (see fig. 1). $P_1 \sqcup Q_1 \parallel P_2 \sqcup Q_2 :\Longleftrightarrow$ There exists $P_3 \sqcup Q_3$ such that

$$(P_3 \sqcup Q_3) \cap (P_1 \cap Q_1) = \emptyset = (P_3 \cap Q_3) \cap (P_2 \sqcup Q_2).$$

Notice that condition (A) is equivalent to: $P_1 \sqcup Q_1 \parallel P_2 \sqcup Q_2$ implies $P_2 \sqcup Q_2 \parallel P_1 \sqcup Q_1$, i.e. parallelism is a symmetric relation. We prove that parallelism is a transitive relation. Suppose $P_1 \sqcup Q_1 \parallel P_2 \sqcup Q_2$ and $P_2 \sqcup Q_2 \parallel P_3 \sqcup Q_3$ (with distinct P_1, P_2, P_3). We prove that $P_1 \sqcup Q_1 \parallel P_3 \sqcup Q_3$.

Case a): $P_1 \parallel Q_1$. Trivial

Case b): $P_1 \nparallel Q_1$, P_1, P_2, $P_3 \in \ell$ for some $\ell \in L$. The transitivity follows at once from the following observation. If c, d, e, $\in C$ and c and d touch in a point P, d and e touch in the same point P, then c and e touch in P. To show this suppose $Q \in c \cap e$, $Q \neq P$, then there are two circles through Q, namely c and e, touching d in P. This contradicts (M6).

Case c): $P_1 \nparallel Q_1$, $P_1 \in [P_2]_\varepsilon$, $P_3 \in [P_2]_{-\varepsilon}$ for some $\varepsilon = +,-$. By definition $P_1 \sqcup Q_1 \parallel P_3 \sqcup Q_3$.

Case d): $P_1 \nparallel Q_1$, $P_1 \parallel_\varepsilon P_2$ for some $\varepsilon = +,-$, $P_3 \nparallel P_1$, $P_3 \nparallel P_2$. Put $P_4 := [P_2]_\varepsilon \cap [P_3]_{-\varepsilon}$. Since $P_2 \sqcup Q_2 \parallel P_3 \sqcup Q_3$ there exists Q_4 such that $P_2 \sqcup Q_2 \parallel P_4 \sqcup Q_4 \parallel P_3 \sqcup Q_3$. Apply case b) to find $P_1 \sqcup Q_1 \parallel P_4 \sqcup Q_4$ and case c) to find $P_1 \sqcup Q_1 \parallel P_3 \sqcup Q_3$.

Case e): $P_1 \nparallel Q_1$, $P_1 \parallel_\varepsilon P_3$ for some $\varepsilon = +,-$, $P_2 \nparallel P_1$, $P_2 \nparallel P_3$. Put $P_4 := [P_1]_\varepsilon \cap [P_2]_{-\varepsilon}$. There exists Q_4 such that $P_1 \sqcup Q_1 \parallel P_4 \sqcup Q_4$ and $P_4 \sqcup Q_4 \parallel P_3 \sqcup Q_3$. Apply case b).

Case f): $P_1 \nparallel Q_1$, P_1, P_2, P_3 mutually nonparallel. Put $P_4 := [P_1]_+ \cap [P_2]_-$. There exists Q_4 and that $P_1 \sqcup Q_1 \parallel P_4 \sqcup Q_4$ and $P_4 \sqcup Q_4 \parallel P_2 \sqcup Q_2$. Apply case d) to find $P_4 \sqcup Q_4 \parallel P_3 \sqcup Q_3$ and so $P_1 \sqcup Q_1 \parallel P_3 \sqcup Q_3$.

Let L^Z be the set of all $P \sqcup Q$, $P, Q \in M_Z$, $P \neq Q$. It is not hard to show that $M^Z := (M_Z, L^Z, \sqcup, \parallel)$ satisfies all the axioms of a nearaffine plane except possibly (P2) or (P2'). For (P2) to hold we have to require:

(B): Let P_1, Q_1, P_2, Q_2 be points as in (A). If $P_1 \in (P_2^+, P_2^-, Q_2)$ and $P_2 \in (P_1^+, P_1^-, Q_1)$. Then circles c and d as described in (A) exist.

If we content ourself with the weaker (P2') we have to require:

(C): Let ε be + or −, A and B two distinct points on $[Z]_\varepsilon$, $A \neq Z \neq B$ and c_1 and c_2 two circles touching in A. Put (see figure 2)

$$C_i := [Z]_{-\varepsilon} \cap c_i, \qquad i = 1,2,$$

$$P_i := [A]_{-\varepsilon} \cap [C_i]_\varepsilon, \qquad i = 1,2,$$

$$Q_i := [B]_\varepsilon \cap c_i, \qquad i = 1,2,$$

$$D_i := [Q_i]_\varepsilon \cap [Z]_{-\varepsilon}, \qquad i = 1,2,$$

$$d_i := (P_i, D_i, B), \qquad i = 1,2.$$

Then d_1 and d_2 touch in B.

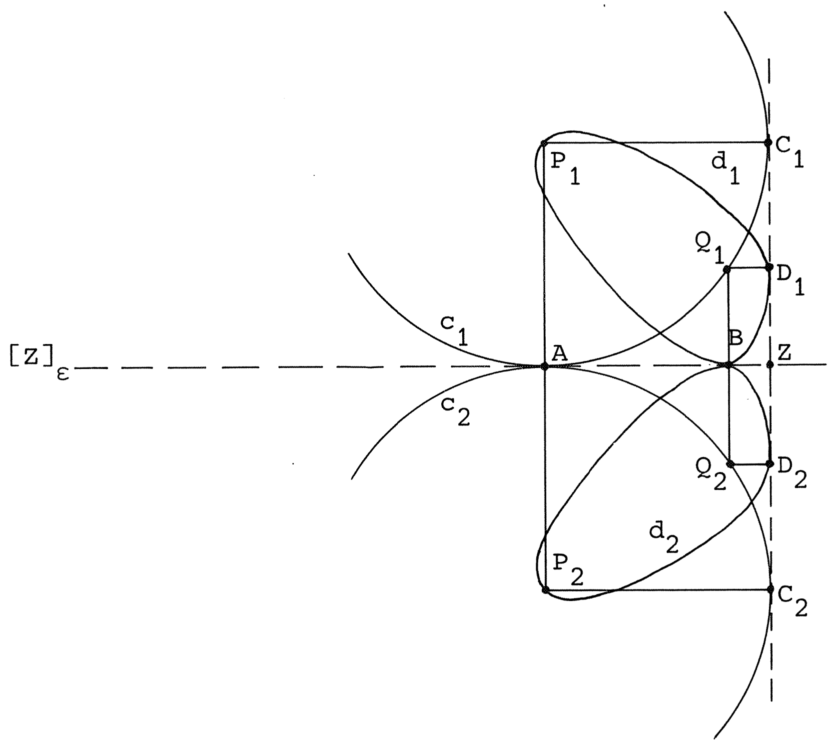

Fig. 2.

If $\mathcal{M}$ is a Minkowski plane satisfying the conditions (A) and (B) or (A) and (C) and Z a point of $\mathcal{M}$, then the nearaffine plane $\mathcal{M}^Z$ is called the *residual plane* with respect to Z.

For the remainder of this section let $\mathcal{M} = (M, \mathcal{L}^+, \mathcal{L}^-, \mathcal{C})$ be a Minkowski plane satisfying the conditions (A) and (C). Since $\sqcup$ and $\parallel$ are defined strictly in terms of the incidence in $\mathcal{M}$ it follows at once that an automorphism of $\mathcal{M}$ fixing a point Z, induces an automorphism of $\mathcal{M}^Z$, i.e. $\mathrm{Aut}(\mathcal{M})_Z \lesssim \mathrm{Aut}(\mathcal{M}^Z)$. In fact, $\mathrm{Aut}(\mathcal{M}_Z) \simeq \mathrm{Aut}(\mathcal{M}^Z)$ as we shall see in a moment. The crucial observation is the following lemma.

3.1. LEMMA. *Let* Z *be a point of* $\mathcal{M}$. *For any two nonparallel points* A *and* B *of* M_Z *let* [A,B] *be the set of points consisting of* A, B, Z *and the points* $C \in M_Z$, *nonparallel to* A *and* B, *for which there is no set* $P \sqcup Q \backslash \{P\}$ *containing* A, B *and* C. *Then*

$$[A,B] = (A,B,Z).$$

PROOF. Clearly both [A,B] and (A,B,Z) contain A, B and Z. Let $C \in (A,B,Z)$, $C \neq A,B,Z$ then $(A,A,C) = (A,B,Z)$. Suppose for some $P,Q \in M_Z$ we have $A,B,C \in P \sqcup Q\backslash\{P\}$. Then $A,B,C \in (P^+,P^-,Q)\backslash\{P^+,P^-\}$, so $(A,B,C) = (P^+,P^-,C)$ a circle not passing through Z, a contradiction. Conversely, let $C \in [A,B]$, $C \neq A,B,Z$ and suppose $C \in (A,B,Z)$. Then $Z \notin (A,B,C)$ and so (A,B,C) intersects $[Z]_+$ and $[Z]_-$ in points P^+ and P^- respectively, different from Z. So, with P defined by $P = [P^+]_- \cap [P^-]_+$, A, B, C are on $P \sqcup Q\backslash\{P\}$, a contradiction. □

The lemma just proved shows that the residual plane $\mathcal{M}^Z$ completely determines the Minkowski plane $\mathcal{M}$. The lines of $\mathcal{M}$ can be recovered from the straight lines of $\mathcal{M}^Z$, the circles not containing Z from the proper lines of $\mathcal{M}^Z$, and the circles containing Z from the sets [A,B]. This proves the following theorem.

3.2. THEOREM. *Let* Y *and* Z *be the points of* $\mathcal{M}$. *Then*

a) $\mathcal{M}^Y \simeq \mathcal{M}^Z$ *iff there exists* $\phi \in \mathrm{Aut}(\mathcal{M})$ *such that* $Y^\phi = Z$.

b) *Any automorphism of* $\mathcal{M}^Z$ *can be extended to an automorphism of* $\mathcal{M}$ *fixing* Z.

c) $\mathrm{Aut}(\mathcal{M})_Z \simeq \mathrm{Aut}(\mathcal{M}^Z)$.

It is not hard to show that for any point Z of M the residual plane M^Z satisfies the Veblen-condition (V'). In fact we can prove somewhat more.

3.3. THEOREM. *Let* $Z \in M$, $\ell \in L$, $\ell \neq [Z]_+, [Z]_-$ *and let* Y *be defined by* $Y = \ell \cap ([Z]_+ \cup [Z]_-)$. *Then*

$$M^Z_{\ell^*} \simeq M_Y,$$

where ℓ^* *is the straight line* $\ell \setminus \{Y\}$ *of* M^Z *(notation as in [9]).*

PROOF. Define an isomorphism $\phi\colon M_Z \to M_Y$ of $M^Z_{\ell^*}$ onto M_Y as follows. For $P \in M_Z$, $P \notin \ell^*$ we define $P^\phi := P$, and for $P \in M_Z$, $P \in \ell^*$, $P^\phi := [P]_{-\varepsilon} \cap [Z]_\varepsilon$, where ε is determined by $\ell \in L^\varepsilon$. □

As a direct consequence of this theorem we have the following result.

3.4. THEOREM. *If the derived plane* M_Z *is a translation plane for every* $Z \in M$, *then the residual plane* M^Z *is a nearaffine translation plane for every* $Z \in M$.

PROOF. Apply 3.3 and 5.2 of [9]. □

As a converse to this theorem we mention the following theorem.

3.5. THEOREM. *Let* Z *be a point of* M. *If* M^Z *is a nearaffine translation plane, then* M_Z *is a translation plane and* M^Z *and* M_Z *have the same translation group.*

PROOF. By 3.2 every automorphism of M^Z is also an automorphism of M_Z, and it is not hard to show that a straight translation of M^Z with a direction corresponding to L^ε is also a translation of M . Let T_+ and T_- be the translation groups of M^Z with directions L^+ and L^- respectively. Since T_+ and T_- are also translation groups of M_Z it follows that T_+ and T_- are elementary abelian. Hence, by 4.12 of [9], the set T of all translation of M^Z is a group and $T = T_+T_-$ = the full translation group of M_Z. □

4. CHARACTERIZATIONS OF THE KNOWN FINITE MODELS

Using the correspondence with sharply triply transitive sets of permutations all known (finite) Minkowski planes can be described as follows. Let p be a prime, h a positive integer, $q := p^h$ and ϕ an automorphism of GF(q). Let $G(\phi)$ be the set of permutations acting on the projective line $\Omega :=$ PG(1,q) = GF(g) $\cup \{\infty\}$ given by

$$x \to \frac{ax+b}{cx+d}, \quad a,b,c,d \in GF(q),\ ad-bc = \text{(nonzero) square in } GF(q),$$

$$x \to \frac{ax^{\phi}+b}{cx^{\phi}+d}, \quad a,b,c,d \in GF(q),\ ad-bc = \text{nonsquare in } GF(q),$$

i.e. $G(\phi) = G_1 \cup \phi\, G_2$, where $G_1 :=$ PSL(2,q) and $G_2 :=$ PG(2,q)\PSL(2,q). Then $G(\phi)$ is sharply triply transitive on Ω (cf. [7], [8], [10]). The residual planes of $(\Omega, G(\phi))$ are easily seen to be the nearaffine translation planes described in [9], section 8. We shall show that a Minkowski plane whose residual planes are nearaffine translation planes, is isomorphic to an $(\Omega, G(\phi))$.

Let c be a circle of a Minkowski plane M of order n and Z a point of M, Z $\notin$ c. If M_Z is augmented to a projective plane, then the points of $c^* = c\backslash([Z]_+ \cup [Z]_-)$ together with the two ideal points corresponding to L^+ and L^- constitute an oval in this projective plane. If n is even, there exists a point (the *nucleus* of the oval) in the projective plane such that the n+1 lines incident with this point are the n+1 tangents of the oval. If n is odd, each point of the projective plane is incident with 0 or 2 tangents (see [3]). From this observation we deduce the following lemma.

<u>4.1. LEMMA</u>. *Let* M *be a Minkowski plane of order* n. *If* n *is even, there cannot exist* 3 *distinct circles* c_1, c_2, d *such that* c_1 *and* c_2 *touch in a point* Z *and* c_i *touches* d *in* $P_i \neq Z$, i = 1,2. *In any case there cannot exist* 4 *distinct circles* c_1, c_2, c_3 *and* d *such that* c_1, c_2, c_3 *touch in a point* Z *and such that* c_i *touches* d *in a point* $P_i \neq Z$, i = 1,2,3.

<u>PROOF</u>. Case n is even. Suppose circles c_1, c_2 and d as described exist. The lines $[[Z]_+ \cap d]_-$ and $[[Z]_- \cap d]_+$ are tangents to the oval corresponding with

d in the projective plane associated with M_Z. They intersect in a point of M_Z. Also c_1 and c_2 are tangents to the oval. They intersect in an ideal point of the projective plane, a contradiction.

Case n is odd. Now c_1, c_2 and c_3 correspond to tangents of the oval d in the projective plane associated with M_Z. They intersect in one (ideal) point, a contradiction. □

4.2. THEOREM. *Let* $M = (\Omega,G) = (M,L^+,L^-,C)$ *be a Minkowski plane of order* $n \geq 5$. *Suppose conditions* (A) *and* (C) *hold in* M *and that* M^Z *is a nearaffine translation plane for every point* Z. *Then* $M \simeq (\Omega,G(\phi))$.

PROOF. Fix $\alpha_1 \in \Omega$. For each point $(\alpha_1,\beta) \in M$ there is an elementary abelian group $T_-(\alpha_1,\beta)$ of translations of $M^{(\alpha_1,\beta)}$ and $M_{(\alpha_1,\beta)}$, and $T_-(\alpha_1,\beta) \lesssim \mathrm{Aut}(M)$ (3.2, 3.4, 3.5). Each $T_-(\alpha_1,\beta)$ fixes all lines of L^- and one line of L^+ (namely the line $\{(\alpha,\beta)\,|\,\alpha \in \Omega\}$). Using the notation of section 2, each $T_-(\alpha_1,\beta)$ consists of positive automorphisms of the form (1,b), where $b \in S^{\Omega}$ fixes β and $Gb = G$, i.e. for each $\beta \in \Omega$ there is an elementary abelian group $B(\beta)$ which fixes β, acts regularly on $\Omega\setminus\{\beta\}$, and for which $GB(\beta) = G$. Define $B := \langle B(\beta)\,|\,\beta \in \Omega\rangle$, then B is doubly transitive on Ω and $GB = G$. Therefore G is a union of cosets of B and in particular $B \subseteq G$. Hence, no nontrivial permutation in B leaves 3 letters fixed. By a theorem of FEIT ([4]), B contains a normal subgroup of order n+1 or there exists an exactly triply transitive permutation group B_0 containing B such that $[B_0:B] \leq 2$. Suppose B contains a normal subgroup of order n+1, then B also contains a sharply doubly transitive subgroup B^*. The circles $\{(\alpha,\alpha^g)\,|\,\alpha \in \Omega\}$, $g \in B^*$ together with the lines $\ell \in L$ now constitute an affine plane of order n+1 and hence configuration as described in 4.1 exist, a contradiction. Therefore $B \leq B_0$, where B_0 is sharply 3-transitive, and $[B_0:B] \leq 2$. All sharply triply transitive groups are known (see [6]). If n is even, then $B_0 \simeq PSL(2,n)$ and so $B = G = PSL(2,n)$, i.e. M is the classical Minkowski plane of order $n = 2^h$. If n is odd, there are at most two sharply 3-transitive groups of degree n+1 and such a group certainly contains PSL(2,n). The Sylow p-subgroups $B(\beta)$ of B are the Sylow p-subgroups of PSL(2,n). Therefore $B \leq PSL(2,n)$ and since $|B| \geq \frac{1}{2}(n+1)(n)(n-1)$ it follows that $B \simeq PSL(2,n)$. Thus, with $G_1 :=$ PSL(2,n) and $G_2 :=$ PSL(2,n)\PSL(2,n),

$$G = G_1 \cup \phi\, G_2$$

for some $\phi \in S^{\Omega}$. It remains to show that ϕ is an automorphism of GF(n). If x,y and z are three distinct points of Ω, then there is a $g \in G_1$ auch that $x^{\phi} = x^g$, $y^{\phi} = y^g$, $z^{\phi} = z^g$ for otherwise there exists $h \in G_2$ such that $x^{\phi} = x^{\phi h}$, $y^{\phi} = y^{\phi h}$, $z^{\phi} = z^{\phi h}$, i.e. h = 1, contradicting $h \in G_2$. If follows that we may assume w.l.o.g. that ϕ fixes 0,1 and ∞. If we do so it also follows that

$$\frac{x^{\phi}-y^{\phi}}{x-y} = \text{square in GF(n) for all } x,y \in GF(n), \quad x \neq y,$$

for $g \in G_1$ determined by $x^{\phi} = x^g$, $y^{\phi} = y^g$, $\infty = \infty^{\phi} = \infty^g$ has determinant $\frac{x^{\phi}-g^{\phi}}{x-y}$. By a theorems of BRUEN and LEVINGER (see [2]) it follows that ϕ is an automorphism of GF(n). □

Using the previous theorem it is possible to give a geometric characterization of the Minkowski planes $(\Omega, G(\phi))$. Consider the following configurational condition:

(D): Let ε be + or −, $\ell \in L^{\varepsilon}$ and V,W two distinct points on ℓ. Suppose c and c' are two distinct circles touching in V. Let Y and Q be two distinct points on c, $Y \not\parallel W$, $Q \not\parallel W$. Define

$$Y' := c' \cap [Y]_{-\varepsilon},$$

$$Q' := c' \cap [Q]_{-\varepsilon},$$

$$d := (Y,Q,W),$$

$$d' := (Y',Q',W).$$

Then d and d' touch in W (see figure 3).

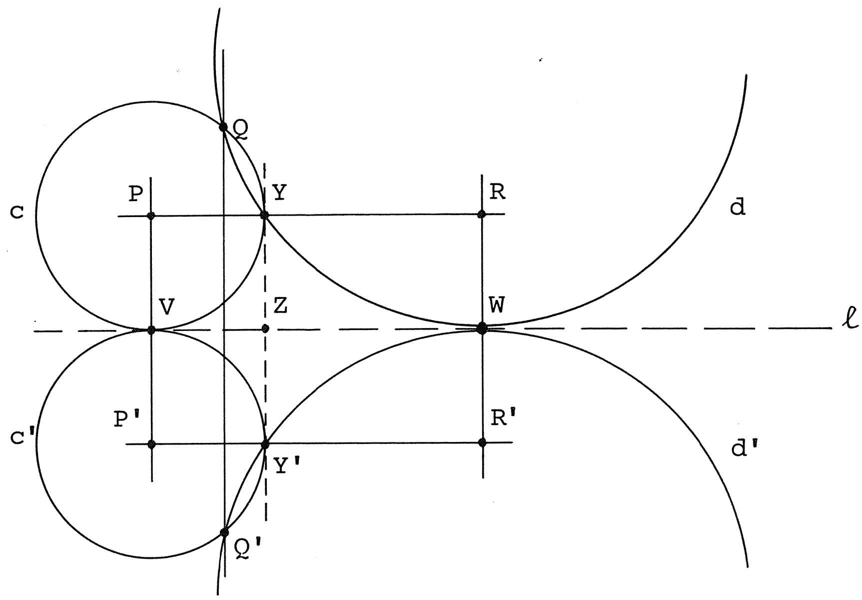

Fig. 3.

Notice that (D) is nothing but a special case of the Desarques configuration (D1) in M^Z on the points P, Q, R, P', Q', R'.

4.3. THEOREM. *Let* M *be a Minkowski plane of order* $n \geq 5$, *and suppose* (D) *holds in* M. *Then* M *is isomorphic to one of the planes* $(\Omega, G(\phi))$.

Of course the proof of 4.3 is based on 4.2 and it is clear that (D) implies (A). Also (C) is a consequence of (D).

4.4. LEMMA. *Let* M *be a Minkowski plane of order* n

a) *If* n *is even then* (A) *implies* (B) *(hence* (C)*).*

b) *In any case* (D) *implies* (C).

PROOF. a) The following statement is easily seen to be equivalent to (B): If the circles c and d as described in (A) exist, then $P_1 \in (P_2^+, P_2^-, Q_2) \Longleftrightarrow P_2 \in (P_1^+, P_1^-, Q_1)$. To prove this last statement, consider the configuration of condition (A) and suppose c and d exist, $P_2 \in (P_1^+, P_1^-, Q_1)$ but $P_1 \notin (P_2^+, P_2^-, Q_2)$. Let e be the circle through P_1 touching (P_2^+, P_2^-, Q_2) and c in P_2^+, f the circle through P_1 touching (P_1^+, P_1^-, Q_1) in P_2. By (A) e and and f touch in P_1. Similarly it follows that the circle g through P_1 touching (P_2^+, P_2^-, Q_2) in P_2 touches f in P_1. Therefore g and e touch in P_1 and so the circles g,e, (P_2^+, P_2^-, Q_2) touch each

other in P_2^+, P_2^-, P_1. This contradicts 4.1 since n is even.
b) Consider the configuration of condition (C). We claim that (P_1,Q_1,Z) and (P_2,Q_2,Z) touch in Z. If (P_i,Q_i,Z) touches c_i in Q_i for i = 1,2, this follows from (A). Suppose therefore that (P_1,Q_1,Z) does not touch c_1 in Q_1, i.e. suppose that (P_1,Q_1,Z) has another point $E_1 \neq Q_1$ in common with c_1. Put $E_2 := [E_1]_{-\varepsilon} \cap c_2$. By (D) the circles (E_2,Q_2,Z) and $(E_1,Q_1,Z) = (P_1,Q_1,Z)$ touch in Z. Suppose (E_2,Q_2,Z) intersects $[A]_{-\varepsilon}$ in a point $P_2' \neq P_2$. Let Y be the point of intersection of $[Z]_\varepsilon$ and (E_2,P_2',C_2). If we apply (D) twice it follows that (E_1,P_1,Y) and (E_1,C_1,Y) both touch (E_2,P_2',C_2) in Y. Hence $(E_1,P_1,Y) = (E_1,C_1,Y)$ and impossibility because $P_1 \| C_1$. We have proved $P_2 \in (E_2,Q_2,Z)$, i.e. (P_1,Q_1,Z) and (P_2,Q_2,Z) touch in Z. So: c_1 and c_2 touch in A implies (P_1,Q_1,Z) and (P_2,Q_2,Z) touch in Z. It is easily seen that the converse also holds. If we replace c_i by d_i, i = 1,2, it follows that d_1 and d_2 touch in B. □

To finish the proof of 4.3 we have to show that all residual planes M^Z are nearaffine translation planes. By 3.4 it suffices to show that all derived planes M_Z are translation planes.

4.5. LEMMA. *Let M be a Minkowski plane satisfying* (D), *then M_Z is a translation plane for every point* Z.

PROOF. Let $Z \in M$ and $P,Q,R,P',Q',R' \in M_Z$ such that $P \|_- P'$, $Q \|_- Q'$, $R \|_- R'$, the line PQ (in M_Z) is parallel to P'Q' and PR is parallel to P'R'. We have to show that QR is parallel to Q'R', i.e. we have to show that the circles (Z,Q,R) and (Z,Q',R') touch in Z. We assume here that P,Q,R (and also P',Q',R') are mutually nonparallel. The other cases follow from the cases we do consider. Put $Y = (P,Q,R) \cap [Z]_+$. If we apply (D) to (P,Q,Z), (P',Q',Z), (P,Q,Y) = (P,Q,R) and (P',Q',Y), it follows that (P,Q,R) and (P',Q',Y) touch in Y. Application of (D) to (P,R,Z), (P',R',Z), (P,R,Y) = (P,Q,R) and (P',R',Y) yields (P,Q,R) and (P',R',Y) touch in Y. Hence (P',Q',Y) = (P',R',Y) = (P',Q',R'). Finally we apply (D) to (Q,R,Y) (Q',R',Y), (Q,R,Z) and (Q',R',Z) and obtain the desired result. □

Notice that it is possible to give a proof of 4.3 without using the theory of nearaffine planes. Show directly, using (D), that any translation of a desired plane M_Z extends to an automorphism of M. Then argue as we did in 4.2.

REFERENCES

1. BENZ, W., *Vorlesungen über Geometrie der Algebren,* Springer-Verlag, Berlin - New York (1973).

2. BRUEN, A. & LEVINGER, B., A theorem on permutations of a finite field, *Can. J. Math.,* 25 (1973), 1060-1065.

3. DEMBOWSKI, P., *Finite geometries,* Springer-Verlag, Berlin - Heidelberg - New York (1968).

4. FEIT, W., On a class of doubly transitive permutation groups, *Ill. J. Math.,* 4 (1960), 170-186.

5. HEISE, W. & KARSEL, H., Symmetrische Minkowski-Ebenen, *J. Geometry,* 3 (1973), 5-20.

6. HUPPERT, B., Scharf dreifach transitive Permutationsgruppen, *Arc. Math.,* 13 (1962), 61-72.

7. PEDRINI, C., 3-reti (non immergibli) aventi dei piani duali di quelli di Moulton quali sotto piani, *Atti Accad. Naz. Lincei Rend. Sc. fis. mat. e nat.,* 40 (1966), 385-392.

8. QUATTROCCHI, P., Sugli insiemi di sostituzioni strettamente 3- transitivi-finiti, *Atti Sem. mat. fis. Univ. Modena,* 24 (1975), 279-289 (1976).

9. WILBRINK, H., Nearaffine planes, To appear.

10. WILBRINK, H., Nearaffine planes and Minkowski planes, Master's thesis, Tech Univ. Eindhoven (1978).

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